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A study of the antisymmetric QRT mappings

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Abstract
We study a particular class of the mappings introduced by Quispel, Roberts and Thompson that come from a two-component system but can be naturally reduced to a one-component one. We classify all these mappings on the basis of the canonical forms of the QRT matrices. We also present the extension of these systems to nonautonomous forms, which are usually discrete Painlevé equations.

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1. Introduction

The QRT mapping, introduced by Quispel, Roberts and Thompson [1] has played a major role in the exploration of integrability in discrete systems. When this family of mappings was discovered the paucity of examples of integrable mappings was extreme. The QRT system not only furnished a testing ground for conjectures related to discrete integrability [2, 3], but also provided the key ingredient for the derivation of discrete Painlevé equations [4]. As a matter of fact, the latter can be obtained from QRT mappings through the process of deautonomization i.e. by letting the parameters of the mappings become functions of the independent variable.

Recently the QRT mapping has been the object of several studies. In [5, 6] the solution of the 'asymmetric' family was presented. The important result is the proof that this solution can be expressed in terms of elliptic functions. In [7] we have shown that the same holds true for another family of mappings which go beyond the QRT family and which possess invariants of higher degree than QRT [8].

In the present study we intend to investigate a particular sub-family which we call 'antisymmetric' for reasons which will become obvious in what follows.

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2. A recall of the QRT mapping and the canonical A matrices

The QRT mapping is presented in two forms, traditionally called 'symmetric' and 'asymmetric'. The key ingredients are two 3×3 matrices, A_0 and A_1

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}. \tag{2.1}$$

If *both* matrices are symmetric the mapping is called symmetric. Otherwise it is called asymmetric. In order to derive the mapping one introduces the vector $\vec{X} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$ and constructs the two vectors $\vec{F} \equiv \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ and $\vec{G} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$ by

$$\vec{F} = (A_0\vec{X}) \times (A_1\vec{X}) \tag{2.2a}$$

$$\vec{G} = (\tilde{A}_0\vec{X}) \times (\tilde{A}_1\vec{X}) \tag{2.2b}$$

where the tilde denotes the transpose of the matrix. The components f_i, g_i of the vectors $\vec{F}, \vec{G}$ are, in general, quartic polynomials of x. In the asymmetric case the mapping assumes the form

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)} \tag{2.3a}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}. \tag{2.3b}$$

In the symmetric case we have $g_i = f_i$ and (2.3) reduces to a single equation

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1} f_2(x_m)}{f_2(x_m) - x_{m-1} f_3(x_m)} \tag{2.4}$$

with the identification $x_n \to x_{2n}, y_n \to x_{2n+1}$. While apparently the mapping involves 18 (resp 12) parameters in the asymmetric (resp symmetric) case, the number of *genuine* parameters is eight for the asymmetric mapping and five for the symmetric one, as explained in [9].

The QRT mapping possesses an invariant which is biquadratic in x and y:

$$(\alpha_0 + K\alpha_1)x_n^2 y_n^2 + (\beta_0 + K\beta_1)x_n^2 y_n + (\gamma_0 + K\gamma_1)x_n^2 + (\delta_0 + K\delta_1)x_n y_n^2 + (\epsilon_0 + K\epsilon_1)x_n y_n + (\zeta_0 + K\zeta_1)x_n + (\kappa_0 + K\kappa_1)y_n^2 + (\lambda_0 + K\lambda_1)y_n + (\mu_0 + K\mu_1) = 0 \tag{2.5}$$

where K plays the role of the integration constant. In the symmetric case the invariant becomes just

$$(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n+1}x_n(x_{n+1} + x_n) + (\gamma_0 + K\gamma_1)\left(x_{n+1}^2 + x_n^2\right) + (\epsilon_0 + K\epsilon_1)x_{n+1}x_n + (\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\mu_0 + K\mu_1) = 0. \tag{2.6}$$

In the symmetric case the invariance of K means simply that $K(x_{n-1}, x_n) = K(x_n, x_{n+1})$. However, for the asymmetric case one should advance one variable at each step i.e. $K(x_n, y_{n-1}) = K(x_n, y_n) = K(x_{n+1}, y_n)$.

In order to construct specific instances of the QRT mapping one must introduce the appropriate A_0 and A_1 matrices. In [5] we have provided a classification of the canonical forms of the A_1 matrices. The idea was to base ourselves on the classification of the forms of discrete Painlevé equations and, at the autonomous limit, construct the corresponding QRT matrices. One can, in fact, choose the A_1 matrix to depend only on the 'family' of the equation and put all the details into the A_0 matrix. The classification we give is based (we must admit, somewhat arbitrarily) on the functional form of the equation, just as we did in [10]. We give below the general form of the equation and the corresponding A_1 matrix

$$\text{(I)}\quad x_{n+1}+x_{n-1}=f(x_n)\qquad A_1=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$\text{(II)}\quad x_{n+1}x_{n-1}=f(x_n)\qquad A_1=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\text{(III)}\quad (x_{n+1}+x_n)(x_n+x_{n-1})=f(x_n)\qquad A_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\text{(IV)}\quad (x_{n+1}x_n-1)(x_nx_{n-1}-1)=f(x_n)\qquad A_1=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$$\text{(V)}\quad \frac{(x_{n+1}+x_n+2z)(x_n+x_{n-1}+2z)}{(x_{n+1}+x_n)(x_n+x_{-1})}=f(x_n)\qquad A_1=\begin{pmatrix}0&0&1\\0&2&2z\\1&2z&0\end{pmatrix}$$

$$\text{(VI)}\quad \frac{(x_{n+1}x_n-z^2)(x_nx_{n-1}-z^2)}{(x_{n+1}x_n-1)(x_nx_{n-1}-1)}=f(x_n)\qquad A_1=\begin{pmatrix}1&0&0\\0&-z^2-1&0\\0&0&z^2\end{pmatrix}$$

$$\text{(VII)}\quad \frac{(x_{n+1}-x_n-z^2)(x_{n-1}-x_n-z^2)+4x_nz^2}{x_{n+1}-2x_n+x_{n-1}-2z^2}=f(x_n)\qquad A_1=\begin{pmatrix}0&0&1\\0&-2&-2z^2\\1&-2z^2&z^4\end{pmatrix}$$

$$\text{(VIII)}\quad \frac{(x_{n+1}z^2-x_n)(x_{n-1}z^2-x_n)-(z^4-1)^2}{(x_{n+1}z^{-2}-x_n)(x_{n-1}z^{-2}-x_n)-(z^{-4}-1)^2}=f(x_n)$$
$$A_1=\begin{pmatrix}0&0&z^4\\0&-z^2(z^4+1)&0\\z^4&0&(z^4-1)^2\end{pmatrix}.$$

The forms of equations presented above correspond to symmetric mappings but they can be extended to asymmetric ones directly, the A_1 matrix being the same. Eight distinct forms of A_1 matrices were thus identified. They will be essential for the derivation of the antisymmetric QRT mappings given in the next section.

3. The antisymmetric QRT mappings

As we have seen in the previous section, the QRT mappings appear either in the 'symmetric', one-component, form or in the 'asymmetric' one, which is written as a system and involves two components. However, there exists a particular class of asymmetric QRT mappings which do not necessitate two components. These mappings are obtained from the general asymmetric case with the restriction that one of the A_i matrices, say A_0 be antisymmetric, while the other, A_1 with the present choice, is symmetric. In this case we have $\vec{G} = -\vec{F}$ and the usual staggering which allows us to reduce (2.3*a, b*) to a single equation (2.4) still works. For obvious reasons we call this mapping, which has the form

$$x_{m+1} = \frac{g_1(x_m) - x_{m-1}g_2(x_m)}{g_2(x_m) - x_{m-1}g_3(x_m)} \tag{3.1}$$

the antisymmetric QRT mapping. While it has the same form as the symmetric QRT, equation (2.4), it is not possible to derive its explicit form from a symmetric choice of A_0, A_1 (generically) and thus it constitutes a new class of mappings having the symmetric QRT appearance. (Another way to obtain a non-standard, one-component QRT mapping is when both A_0 and A_1 are antisymmetric. However, this case is not very interesting. The only resulting mapping is $x_{n+1}x_{n-1} - (x_{n+1} + x_{n-1})x_n + x_n^2 = 0$ which factorizes into $(x_{n+1} - x_n)(x_{n-1} - x_n) = 0$ and thus does not define a nontrivial evolution).

We start with the generic antisymmetric A_0 matrix:

$$A_0 = \begin{pmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \zeta \\ -\gamma & -\zeta & 0 \end{pmatrix}. \tag{3.2}$$

In order to derive the antisymmetric QRT mapping we combine the antisymmetric A_0 with the canonical A_1 given in section 2. We complement this derivation by deautonomizing the resulting mappings, obtaining equations usually in the discrete Painlevé class.

(I) Taking $\beta = 1$, (since $\beta = 0$ leads to a trivial mapping), we find

$$x_{n+1} + x_{n-1} = \frac{x_n^2 - \zeta}{x_n + \gamma} \tag{3.3}$$

and translating x we find finally

$$x_{n+1} + x_{n-1} = x_n + \frac{z}{x_n}. \tag{3.4}$$

We deautonomize this mapping using either of the discrete integrability criteria which are perfectly suitable for the present case, singularity confinement or algebraic entropy. The application of these criteria to the mapping (3.4) yields the following n-dependence for z: $z_n = pn + r + s(-1)^n$. Since we are restricting ourselves to mappings which have a 'symmetric' appearance, we discard the even–odd-dependent term $s(-1)^n$, and find simply $z_n = pn + r$. Changing the sign of every other x we find

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + t \tag{3.5}$$

in the special case $t = 0$. Equation (3.5) with generically non-zero t and s is exactly the equation we called the asymmetric d-$\mathrm{P_I}$ which is just the contiguity relation of the solutions of $\mathrm{P_{IV}}$ [11], t being the continuous variable of the latter.

(II) This case leads to the mapping (with $\gamma = 1$, lest the equation becomes trivial)

$$x_{n+1}x_{n-1} = -\frac{x_n(x_n - \zeta)}{\beta x_n + 1} \tag{3.6}$$

which, though very similar, is not of the symmetric QRT type because of the presence of the minus sign on the right-hand side. The deautonomization of this mapping leads to $\zeta = \zeta_0\lambda^n, \beta = \beta_0\lambda^n$ (if we neglect all even–odd dependences). The resulting mapping is equivalent to a known Painlevé equation. Indeed if we define $x_{2n} = -1/(\beta_{2n}y_{2n})$ and $x_{2n+1} = \zeta_{2n+1}y_{2n+1}$ we obtain a symmetric form:

$$y_{n+1}y_{n-1} = \frac{1}{\beta_n^2\zeta_n^2}\frac{1 + \beta_n\zeta_n y_n}{y_n(1 - y_n)}. \tag{3.7}$$

This is a special case of an equation derived in [12] and studied by Kajiwara *et al* in [13].

Taking $\beta = 0$ in (3.6) we find the mapping $x_{n+1}x_{n-1} = -x_n(x_n - \zeta)$. In this case, the n-dependence of ζ can be absorbed through a gauge transformation. The solution of this autonomous equation can be expressed in terms of hyperbolic functions. This is in perfect analogy to the case of the continuous $\mathrm{P_{III}}$ equation $w'' = w'^2/w - w'/t + \beta/t + \delta/w$ when the two other terms are absent. In that case the equation can be transformed into an autonomous one and solved in terms of hyperbolic functions.

(III) In the next case we obtain the mapping

$$(x_n + x_{n+1})(x_n + x_{n-1}) = \frac{2x_n\left(x_n^2 - \zeta\right)}{x_n + \gamma}. \tag{3.8}$$

The deautonomization of this mapping leads to constant ζ and $\gamma = pn + r$, if one restricts oneself to the symmetric case. This mapping is an extremely curious one, and has never been identified to this day. If we reintroduce the full freedom, we have an asymmetric mapping:

$$(x_n + x_{n+1})(x_n + x_{n-1}) = \frac{1}{t}\frac{(x_n - a)(x_n - b)(x_n - c)}{x_n + z_n} \tag{3.9a}$$

$$(x_{n+2} + x_{n+1})(x_n + x_{n+1}) = \frac{1}{1-t}\frac{(x_{n+1} + a)(x_{n+1} + b)(x_{n+1} + c)}{x_{n+1} + z_{n+1}} \tag{3.9b}$$

with $z_n = pn + r + s(-1)^n$ and equation (3.8) corresponds to $a = s = 0, c = -b, t = 1/2$. This equation is a contiguity relation of the Ablowitz–Fokas equation [14] and will be studied in more detail elsewhere [15].

When $\gamma = 0$, the mapping reduces to $(x_n + x_{n+1})(x_n + x_{n-1}) = 2\left(x_n^2 - \zeta\right)$. This is a special case of the mapping $(x_n + x_{n+1})(x_n + x_{n-1}) = f(n)\left(x_n^2 - \zeta\right)$ obtained in [10] where it was shown to be linearizable. On the other hand taking $\zeta = \gamma^2$ leads to the degenerate (in the sense introduced in [10]) form

$$(x_n + x_{n+1})(x_n + x_{n-1}) = 2x_n(x_n - \gamma). \tag{3.10}$$

The deautonomization of (3.10) leads to $\gamma = pn + r$, if one restricts oneself to the symmetric case. This equation is deeply related to (3.8) and the full case will be studied in detail in [15].

(IV) Here we find the mapping

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = -\frac{x_n^4 - 1 + (\beta + \zeta)x_n\left(x_n^2 - 1\right)}{\beta x_n + 1}. \tag{3.11}$$

In order to deautonomize this mapping it is convenient to rewrite it as

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = -\frac{\left(x_n^2 - 1\right)(x_n - \alpha)(x_n - 1/\alpha)}{\beta x + 1}. \tag{3.12}$$

We found that the deautonomization satisfying the integrability criteria leads to $\alpha =$ constant, $\beta_n = \beta_0\lambda^n$. This is a special case of the q-discrete Ablowitz–Fokas equation we studied in [16, 17].

Degenerate cases can also be obtained. Taking $\beta = -1/\alpha$ gives the mapping

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \left(1 - x_n^2\right)(1 - \alpha x_n) \tag{3.13}$$

and its deautonomization results in $\alpha_n = \alpha_0\lambda^n$ (provided we restrict ourselves to purely symmetric terms). This is deeply related to the q-discrete Ablowitz–Fokas equation.

Another degenerate case can be obtained with $\beta = 1$, leading to

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = (1 - x_n)(x_n - \alpha)(x_n - 1/\alpha) \tag{3.14}$$

which is simply a periodic one: $x_{n+4} = x_n$.

(V) Here, for the generic case $\beta \neq 0$, we find the mapping (with $\beta = 1$)

$$\left(\frac{x_n + x_{n+1} + 2z}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + 2z}{x_n + x_{n-1}}\right) = \frac{(x_n + z)(x_n(x_n + 2z) + 2\gamma z - \zeta)}{x_n\left(x_n^2 - \zeta\right)}. \tag{3.15}$$

For the purpose of deautonomizing we rewrite (3.15) as

$$\left(\frac{x_n + x_{n+1} + z_n + z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + z_n + z_{n-1}}{x_n + x_{n-1}}\right) = \frac{(x_n + z_n)((x_n + z_n)^2 - a^2)}{x_n\left(x_n^2 - b^2\right)} \tag{3.16}$$

where, of course, prior to deautonomization, $z_n = z_{n+1} = z_{n-1}$. Restricting ourselves to deautonomizations that do not introduce asymmetries we find $z_n = pn + r$ while a and b are constants.

Taking $a \to \infty$ and $b \to \infty$ with ratio $a/b = c$ we obtain the mapping

$$\left(\frac{x_n + x_{n+1} + z_n + z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + z_n + z_{n-1}}{x_n + x_{n-1}}\right) = c^2\frac{x_n + z_n}{x_n} \tag{3.17}$$

which is not a discrete Painlevé equation but rather a linearizable one. Its linearization is straightforward. We take $cu_n = (x_n + x_{n+1} + z_n + z_{n+1})/(x_n + x_{n+1})$ which leads to

$$\frac{z_{n+1}}{u_n u_{n+1} - 1} + \frac{z_n}{u_n u_{n-1} - 1} = \frac{z_{n+1} + z_n}{cu_n - 1}. \tag{3.18}$$

It suffices now to put $u = (cw - 1)/(w - c)$ in order to transform (3.18) into

$$\frac{z_{n+1}}{w_n w_{n+1} - 1} + \frac{z_n}{w_n w_{n-1} - 1} = 0 \tag{3.19}$$

with obvious solution. From (3.19) it is clear that z can be *any* function of n and not just linear.

Next we examine the degenerate cases of (3.16). Taking $b = z$ we obtain the mapping

$$\left(\frac{x_n + x_{n+1} + z_n + z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + z_n + z_{n-1}}{x_n + x_{n-1}}\right) = \frac{(x_n + z_n)^2 - a^2}{x_n(x_n - z_n)} \tag{3.20}$$

while the choice $a = z$ leads to the dual form

$$\left(\frac{x_n + x_{n+1} + z_n + z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + z_n + z_{n-1}}{x_n + x_{n-1}}\right) = \frac{(x_n + z_n)(x_n + 2z_n)}{x_n^2 - b^2}. \tag{3.21}$$

In both cases the deautonomization, preserving the one-component form, gives $z_n = pn + r$ with a and b constant in (3.20) and (3.21) respectively. Finally we perform the double degeneracy $a = b = z$, which results in the mapping

$$\left(\frac{x_n + x_{n+1} + z_n + z_{n+1}}{x_n + x_{n+1}}\right)\left(\frac{x_n + x_{n-1} + z_n + z_{n-1}}{x_n + x_{n-1}}\right) = \frac{x_n + 2z_n}{x_n - z_n} \tag{3.22}$$

again with $z_n = pn + r$ as nonautonomous extension.

(VI) Again we start from the generic case $\gamma \neq 0$ and find the mapping (for $\gamma = 1$)

$$\left(\frac{x_n x_{n+1} - z^2}{x_n x_{n+1} - 1}\right)\left(\frac{x_n x_{n-1} - z^2}{x_n x_{n-1} - 1}\right) = \frac{\left(x_n^2 - z^2\right)(x_n + bz)(x_n + z/b)}{\left(x_n^2 - 1\right)(x_n + a)(x_n + 1/a)}. \tag{3.23}$$

Again, in view of the deautonomization, we rewrite (3.23) as

$$\left(\frac{x_n x_{n+1} - z_n z_{n+1}}{x_n x_{n+1} - 1}\right)\left(\frac{x_n x_{n-1} - z_n z_{n-1}}{x_n x_{n-1} - 1}\right) = \frac{\left(x_n^2 - z_n^2\right)(x_n + bz_n)(x_n + z_n/b)}{\left(x_n^2 - 1\right)(x_n + a)(x_n + 1/a)} \tag{3.24}$$

and obtain, by applying discrete integrability criteria while excluding asymmetric cases, $z_n = z_0\lambda^n$ and a, b constant. A limiting case can be obtained by taking $a \to 0$ and $b \to 0$ with ratio $a/b = c^2$,

$$\left(\frac{x_n x_{n+1} - z_n z_{n+1}}{x_n x_{n+1} - 1}\right)\left(\frac{x_n x_{n-1} - z_n z_{n-1}}{x_n x_{n-1} - 1}\right) = c^2 z_n \frac{x_n^2 - z_n^2}{x_n^2 - 1}. \tag{3.25}$$

Performing the change of variable $c\sqrt{q_n}u_n = (x_nx_{n+1} - z_nz_{n+1})/(x_nx_{n+1} - 1)$ where $q_n = (z_nz_{n+1})^{1/2}$ we find

$$\left(\frac{u_nu_{n+1} - q_nq_{n+1}}{u_nu_{n+1} - 1}\right)\left(\frac{u_nu_{n-1} - q_nq_{n-1}}{u_nu_{n-1} - 1}\right) = \left(\frac{u_n - q_n^{3/2}/c}{u_n - q_n^{-1/2}/c}\right)^2 \tag{3.26}$$

which is a special case of an equation already obtained in [18].

Degenerate cases can be also derived from (3.23). The choice $a = z$ leads to

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n - z_n)(x_n + bz_n)(x_n + z_n/b)}{(x_n^2 - 1)(x_n + 1/z_n)} \tag{3.27}$$

where, again, excluding asymmetric cases, $z_n = z_0\lambda^n$ and b constant. A dual case ($b = z$, a constant) leads to

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n^2 - z_n^2)(x_n + z_n^2)}{(x_n - 1)(x_n + a)(x_n + 1/a)}. \tag{3.28}$$

On the other hand one can also cancel one a factor with a b factor, for instance taking $b = az$ to the formal form

$$\left(\frac{x_nx_{n+1} - z^2}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z^2}{x_nx_{n-1} - 1}\right) = \frac{(x_n^2 - z^2)(x_n + az^2)}{(x_n^2 - 1)(x_n + a)}. \tag{3.29}$$

The n-dependence is more subtle in this case as neither a nor b are constants. In fact the discrete integrability criteria lead (excluding asymmetry) to

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n^2 - z^2)(x_n + cz_n^{3/2})}{(x_n^2 - 1)(x_n + cz_n^{-1/2})} \tag{3.30}$$

with $z_n = z_0\lambda^n$ and c constant. Taking $c \to 0$ or $c \to \infty$ we can obtain two limits of (3.30) of the form

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = h_n\frac{x_n^2 - z_n^2}{x_n^2 - 1} \tag{3.31}$$

where h_n is equal to 1 or z_n^2 respectively.

Further degeneracies can be obtained. Taking $a = z^2$ in (3.28) leads to

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n^2 - z_n^2)}{(x_n - 1)(x_n + 1/z_n^2)} \tag{3.32}$$

while a dual form of the latter can be obtained from (3.27) with $b = z^2$.

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n - z_n)(x_n + z_n^3)}{(x_n^2 - 1)} \tag{3.33}$$

and a self-dual form can be obtained from either (3.26) or (3.27)

$$\left(\frac{x_nx_{n+1} - z_nz_{n+1}}{x_nx_{n+1} - 1}\right)\left(\frac{x_nx_{n-1} - z_nz_{n-1}}{x_nx_{n-1} - 1}\right) = \frac{(x_n - z_n)(x_n + z_n^2)}{(x_n - 1)(x_n + 1/z_n)}. \tag{3.34}$$

Finally we have the mappings corresponding to the two cases VII and VIII. The autonomous forms are given below. From VII with $\beta = 1$ we have

$$\frac{(x_{n+1} - x_n - z^2)(x_{n-1} - x_n - z^2) + 4x_nz^2}{x_{n+1} - 2x_n + x_{n-1} - 2z^2} = -z^2\frac{5x_n^2 + (z^2 + 6\gamma)x_n + \gamma z^2 + \zeta}{x_n^2 + 2(z^2 + \gamma)x_n + 2\gamma z^2 + \zeta}. \tag{3.35}$$

Limiting cases can be calculated in a straightforward way. Similarly for VIII with $\gamma = 1/(z^2+1)$ we have

$$\frac{(x_{n+1}z^2 - x_n)(x_{n-1}z^2 - x_n) - (z^4-1)^2}{(x_{n+1}z^{-2} - x_n)(x_{n-1}z^{-2} - x_n) - (z^{-4}-1)^2} = -z^8 \frac{\beta x_n^3 + x_n^3 + (\zeta z^2 + \beta(z^2+1)^2(z^2-1))x_n + z^4 - 1}{\beta z^6 x_n^3 + z^4 x_n^3 + (\zeta z^4 - \beta(z^2+1)^2(z^2-1))x_n - z^4 + 1}. \tag{3.36}$$

We shall not attempt any deautonomization of these two cases. As a matter of fact the study of the discrete Painlevé equations which come from these two classes [19] shows that one would have had not only to allow z and the parameters to depend on n but also to transform substantially the left-hand side of (3.35) and (3.36).

4. Two special cases

In the previous section we have examined the antisymmetric QRT mappings, as well as their deautonomizations, coming from the eight classes of canonical A_1 matrices. However, there exist two familiar forms of discrete Painlevé equations the autonomous forms of which cannot be directly read-off from the mappings associated with these eight classes. Indeed they correspond to some limit and in particular to $z \to 0$ in class V and $z \to 1$ in class VI. We have thus

$$(\widetilde{\mathrm{V}}) \qquad \frac{1}{x_{n+1}+x_n} + \frac{1}{x_n + x_{n-1}} = f(x_n) \qquad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(\widetilde{\mathrm{VI}}) \qquad \frac{1}{x_{n+1}x_n - 1} + \frac{1}{x_n x_{n-1} - 1} = f(x_n) \qquad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The same limits in the case of classes VII and VIII lead to, essentially, the same A_1 matrices thus there is no need to study them apart.

In the case of class $(\widetilde{\mathrm{V}})$ we find the mapping

$$\frac{1}{x_{n+1}+x_n} + \frac{1}{x_n + x_{n-1}} = \frac{3\beta x_n^2 + 2\gamma x_n - \zeta}{2x_n\left(\beta x_n^2 - \zeta\right)}. \tag{4.1}$$

For the purpose of deautonomization we must introduce the proper n-dependence and the best way to do this is to take consistently the limit $z \to 0$ in (3.16). We find thus

$$\frac{z_{n+1}+z_n}{x_{n+1}+x_n} + \frac{z_n + z_{n-1}}{x_n + x_{n-1}} = \frac{z_n\left(3x_n^2 - b\right) - ax_n}{x_n\left(x_n^2 - b\right)} \tag{4.2}$$

and $z_n = pn + q$. Interesting limits of (4.2) do exist. Taking $b \to \infty$ and also $a \to \infty$, with $a/b = c$ we find the equation

$$\frac{z_{n+1}+z_n}{x_{n+1}+x_n} + \frac{z_n + z_{n-1}}{x_n + x_{n-1}} = \frac{z_n}{x_n} + c \tag{4.3}$$

which has been identified in [20] as a being linearizable for any function z of n. Taking $b = 0$ we find

$$\frac{z_{n+1}+z_n}{x_{n+1}+x_n} + \frac{z_n + z_{n-1}}{x_n + x_{n-1}} = \frac{3z_n}{x_n} - \frac{a}{x_n^2} \tag{4.4}$$

which is also a linearizable equation. In fact for z to be a free function of n we rewrite the equation as

$$\frac{z_{n+1}+z_n}{x_{n+1}+x_n}+\frac{z_n+z_{n-1}}{x_n+x_{n-1}}=\frac{z_{n+1}+z_n+z_{n-1}}{x_n}-\frac{a}{x_n^2}. \tag{4.5}$$

It suffices now to invert x, i.e. $x \to 1/x$, and, after some elementary manipulations, the equation reduces to precisely (4.3) with $c = a$.

In the case of ($\widetilde{\mathrm{VI}}$) we find

$$\frac{1}{x_{n+1}x_n-1}+\frac{1}{x_nx_{n-1}-1}=-\frac{\beta x_n\left(x_n^2-2\right)-\zeta x_n-2\gamma}{\left(x_n^2-1\right)\left(\gamma\left(x_n^2+1\right)+(\beta+\zeta)x_n\right)}. \tag{4.6}$$

Again for the correct deautonomization of (4.6) we must start from (3.24) and implement consistently the limit $z \to 1$, while taking $b \to a$. We obtain thus

$$\frac{z_n+z_{n+1}}{x_nx_{n+1}-1}+\frac{z_n+z_{n-1}}{x_nx_{n-1}-1}=\frac{cx_n\left(x_n^2-3\right)z_n+dx_n\left(x_n^2-1\right)+4z_n}{\left(x_n^2-1\right)\left(x_n^2+cx_n+1\right)} \tag{4.7}$$

where the fact that the *same* constant c appears in both the numerator and the denominator of (4.7) is imposed by the autonomous limit. We find that the deautonomization of (4.7) satisfying discrete integrability criteria leads to $z_n = pn + q$. Limiting cases of (4.7) do exist. If we take $c \to \infty$ and $d \to \infty$ such that $d/c = k$ we find the mapping

$$\frac{z_n+z_{n+1}}{x_nx_{n+1}-1}+\frac{z_n+z_{n-1}}{x_nx_{n-1}-1}=z_n\frac{x_n^2-3}{x_n^2-1}+k \tag{4.8}$$

while when d remains finite we find (4.8) with $k = 0$.

5. Conclusions

In this paper we have examined a special class of QRT mappings which create a (minor) paradox. They are integrable mappings which have the symmetric QRT form but do not belong to the symmetric QRT family. The analysis we presented here showed that such mappings can be obtained from a generic A_1 matrix and an *antisymmetric* A_0, i.e. a matrix of the form (3.2). Thus the correct interpretation of these mappings is that they belong to the *asymmetric* QRT family and just happen to have a symmetric form due to the special form of A_0. We have dubbed these mappings 'antisymmetric'.

It is interesting to point out that the converse situation does exist. Indeed, there exist integrable mappings of the asymmetric QRT form, i.e. mappings involving two components, which do not really belong to this class but which are in fact *symmetric* QRT mappings. Let us present such an example here. We start with a mapping of the class (VI) belonging to the symmetric QRT family. Its generic form is then

$$\left(\frac{x_nx_{n+1}-z^2}{x_nx_{n+1}-1}\right)\left(\frac{x_nx_{n-1}-z^2}{x_nx_{n-1}-1}\right)=\frac{\left(x_n^2+ax_n+z^2\right)\left(x_n^2+bx_n+z^2\right)}{\left(x_n^2+cx_n+1\right)\left(x_n^2+dx_n+1\right)}. \tag{5.1}$$

Putting $y = (x_nx_{n+1} - z^2)/(x_nx_{n+1} - 1)$ we can rewrite (5.1) as

$$x_nx_{n+1}=\frac{y_n-z^2}{y_n-1} \tag{5.2a}$$

$$y_ny_{n-1}=\frac{\left(x_n^2+ax_n+z^2\right)\left(x_n^2+bx_n+z^2\right)}{\left(x_n^2+cx_n+1\right)\left(x_n^2+dx_n+1\right)}. \tag{5.2b}$$

This equation is indeed of asymmetric QRT form but does not belong to that family. As a matter of fact the asymmetric QRT mappings of class (II) may have right-hand sides which are ratios of quadratic polynomials but not quartic. Thus (5.2) is not an acceptable form. However, this class of equation does not warrant a special study. Given that (5.2a) has a homographic rhs, it is clear that the first transformation one would attempt would be to eliminate y, reducing the mapping to a symmetric QRT one and thus solving the paradox.

References

[1] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183
[2] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[3] Hietarinta J and Viallet C 1998 *Phys. Rev. Lett.* **81** 325
[4] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829
[5] Ramani A, Carstea S, Grammaticos B and Ohta Y 2002 *Physica* A **305** 437
[6] Iatrou A and Roberts J A G 2001 *J. Phys. A: Math. Gen.* **34** 6617
[7] Viallet C M, Ramani A and Grammaticos B 2004 *Phys. Lett.* A **322** 186
[8] Kimura K, Yahagi H, Hirota R, Ramani A, Grammaticos B and Ohta Y 2002 *J. Phys. A: Math. Gen.* **35** 9205
[9] Grammaticos B and Ramani A 1997 *Methods Appl. An.* **4** 196
[10] Ramani A and Grammaticos B 1996 *Physica* A **228** 160
[11] Ramani B and Grammaticos A 1998 *J. Phys. A: Math. Gen.* **31** 5787
[12] Kruskal M D, Tamizhmani K M, Grammaticos B and Ramani A 2000 *Reg. Chaot. Dyn.* **5** 273
[13] Kajiwara K and Kimura K 2003 *J. Nonlinear Math. Phys.* **10** 86
[14] Fokas A S and Ablowitz M J 1982 *J. Math. Phys.* **23** 2033
[15] Ramani A, Grammaticos B and Willox R in preparation
[16] Jimbo M, Sakai H, Ramani A and Grammaticos B 1996 *Phys. Lett.* A **217** 111
[17] Ramani A and Grammaticos B 2005 *Chaos Solitons Fractals* **24** 1331
[18] Ramani A, Willox R, Grammaticos B, Carstea A S and Satsuma J 2005 *Physica* A **347** 1
[19] Ohta Y, Ramani A and Grammaticos B 2001 *J. Phys. A: Math. Gen.* **34** 10523
[20] Ramani A, Grammaticos B and Tamizhmani T 2004 On the alternate discrete Painlevé equations and related systems *Preprint*